Non-perturbative Renormalization for B_K

Hwancheol Jeong, Jangho Kim, Weonjong Lee, Jeonghwan Pak, Sungwoo Park

Lattice Gauge Theory Research Center
Department of Physics and Astronomy
Seoul National University

KIAS Workshop 2014

Introduction of NPR for Four-Fermion Operators

- We present matching factors for the four-fermion operators relevant to B_K obtained using the non-perturbative renormalization (NPR) method in the RI-MOM scheme
- We compare NPR results with those of one-loop perturbative matching.

Momentum in Reduced Brillouin Zone

- $\tilde{p}$ is the momentum in reduced Brillouin zone.

$$p \in (-\frac{\pi}{a}, \frac{\pi}{a}]^4, \qquad \tilde{p} \in (-\frac{\pi}{2a}, \frac{\pi}{2a}]^4, \qquad p = \tilde{p} + \pi_B$$

  where $\pi_B (\equiv \frac{\pi}{a} B)$ is cut-off momentum in hypercube.
- a : lattice spacing.
- B : vector in hypercube. Each element is 0 or 1
  ex) $B = (0, 0, 1, 1)$

One-color Trace and Two-color Trace

The one-color trace four-fermion operator

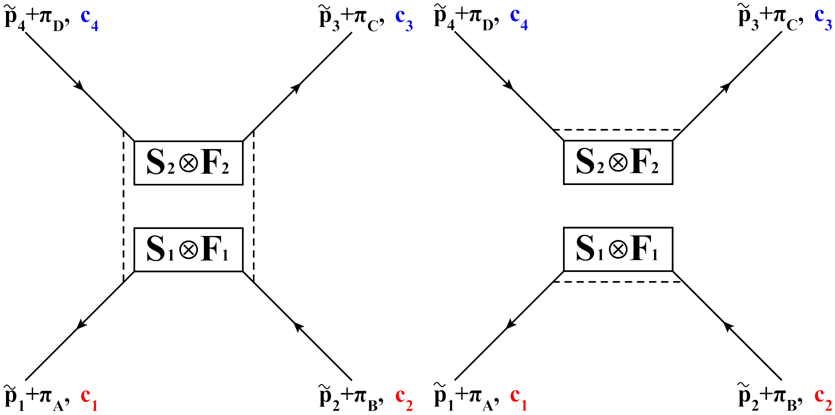
$$O_{\mathbf{I}}(x) = \left[\overline{\chi}_{c_1}(x_A)\overline{(\gamma_{S_1} \otimes \xi_{F_1})}_{AB}\chi_{c_2}(x_B)\right]\left[\overline{\chi}_{c_3}(x_C)\overline{(\gamma_{S_2} \otimes \xi_{F_2})}_{CD}\chi_{c_4}(x_D)\right] \times U_{AD;c_1 c_4}(x) U_{BC;c_2 c_3}(x)$$

The two-color trace four-fermion operator

$$O_{\mathbf{II}}(x) = \left[\overline{\chi}_{c_1}(x_A)\overline{(\gamma_{S_1} \otimes \xi_{F_1})}_{AB}\chi_{c_2}(x_B)\right]\left[\overline{\chi}_{c_3}(x_C)\overline{(\gamma_{S_2} \otimes \xi_{F_2})}_{CD}\chi_{c_4}(x_D)\right] \times U_{AB;c_1 c_2}(x) U_{CD;c_3 c_4}(x)$$

- U : link variable.
- A, B, C, D: hypercube index
- c_1, c_2, c_3, c_4 : color index

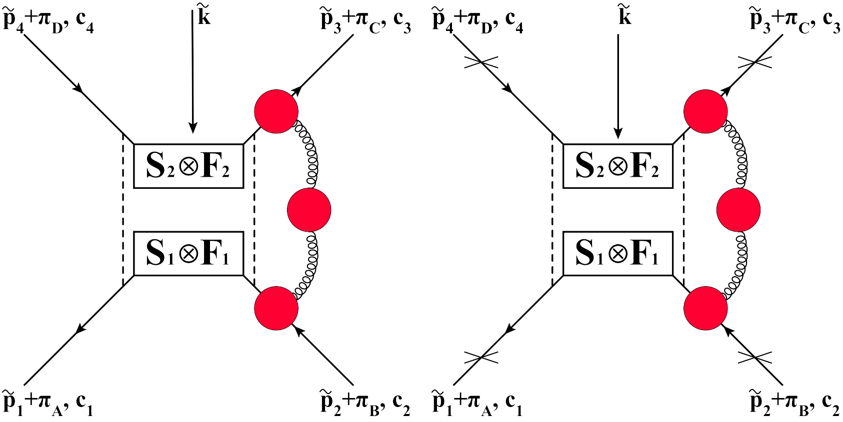
One-color Trace and Two-color Trace



(a) one-color trace (b) two-color trace

One-color trace and two-color trace four-fermion operators

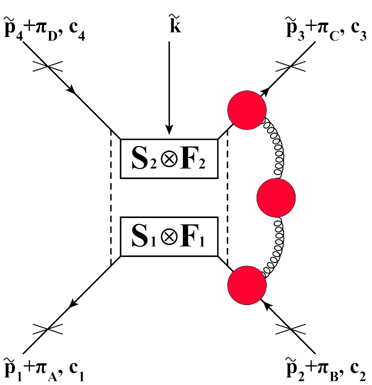
Amputated Green's Function



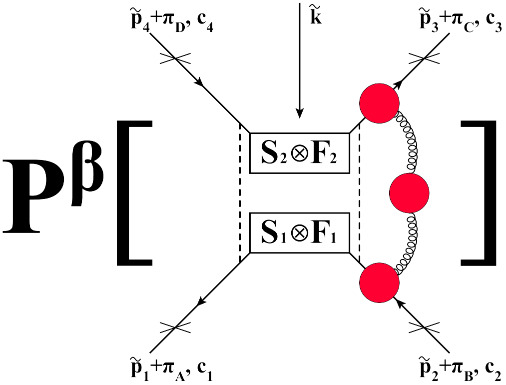
(a) Unamputated Green's Function (b) Amputated Green's Function

Red Circle : 1PI Diagram.

Projected Amputated Green's Function



(c) Amputated Green's Function : Λ^{α}



(d) Projected Amputated Green's Function : $\Gamma^{\alpha\beta}$

- α, β : the indices to represent different operators.
  ex) $\alpha = (V \otimes P)(V \otimes P)_{\mathbf{I}}$, $\beta = (A \otimes P)(A \otimes P)_{\mathbf{II}}$
- Momentum is conserved in reduced Brillouin zone. $\widetilde{k} = \widetilde{p}_1 - \widetilde{p}_2 + \widetilde{p}_3 - \widetilde{p}_4$

Projection Operators

The one-color trace projection operator is

$$\hat{P}^{\beta}_{\mathbf{I}} = \frac{1}{N}\overline{\overline{(\gamma^{\dagger}_{S'} \otimes \xi^{\dagger}_{F'})}}_{BA}\overline{\overline{(\gamma^{\dagger}_{S''} \otimes \xi^{\dagger}_{F''})}}_{DC}\delta_{c_4 c_1}\delta_{c_3 c_2}$$

The two-color trace projection operator is

$$\hat{P}^{\beta}_{\mathbf{II}} = \frac{1}{N}\overline{\overline{(\gamma^{\dagger}_{S'} \otimes \xi^{\dagger}_{F'})}}_{BA}\overline{\overline{(\gamma^{\dagger}_{S''} \otimes \xi^{\dagger}_{F''})}}_{DC}\delta_{c_2 c_1}\delta_{c_4 c_3}$$

- N $= 3072 = \underbrace{4^4}_{\text{spin}} \times \underbrace{4^4}_{\text{taste}} \times (\underbrace{3}_{\text{1-color tr}} + \underbrace{9}_{\text{2-color tr}})$.
- A, B, C, D: hypercube index
- c_1, c_2, c_3, c_4 : color index

For simplicity, we define the following notations.

$$O_1 \equiv O_{[V\otimes P][V\otimes P],\mathbf{I}}$$
$$O_2 \equiv O_{[V\otimes P][V\otimes P],\mathbf{II}}$$
$$O_3 \equiv O_{[A\otimes P][A\otimes P],\mathbf{I}}$$
$$O_4 \equiv O_{[A\otimes P][A\otimes P],\mathbf{II}}$$

The tree level B_K operator is

$$O_{B_K}^{\text{tree}} = O_1^{\text{tree}} + O_2^{\text{tree}} + O_3^{\text{tree}} + O_4^{\text{tree}}$$

The matching formula of B_K operator is

$$O_{B_K}^{R} = z_1 O_1^B + z_2 O_2^B + z_3 O_3^B + z_4 O_4^B + \sum_{\alpha\in(D)} z_\alpha O_\alpha^B$$

- The superscript R(B) denotes renormalized(bare) quantity.
- z_1, z_2, z_3, z_4 and z_α are the renormalization factors.
- We classify operators as follows.
  (C): $\{O_1, O_2, O_3, O_4\}$ (diagonal operators)
  (D): remaining operators (off-diagonal operators)

The renormalization of quark field is

$$\chi_R = Z_q^{1/2} \chi_B$$

where z_q is wave function renormalization factor for quark field.
Multiplying the inverse propagators to the unamputated Green's function, we obtain the amputated Green's function Λ.
The renormalized amputated Green's function of B_K operator is

$$\Lambda^R_{B_K} = \frac{z_1}{z_q^2}\Lambda^B_1 + \frac{z_2}{z_q^2}\Lambda^B_2 + \frac{z_3}{z_q^2}\Lambda^B_3 + \frac{z_4}{z_q^2}\Lambda^B_4 + \sum_{\alpha \in (D)} \frac{z_\alpha}{z_q^2}\Lambda^B_\alpha ,$$

For simplicity, the projection operators are defined as follows.

$$\hat{P}_1 = \frac{1}{N} \overline{\overline{(V_\mu \otimes P)}}^{\dagger}_{BA} \overline{\overline{(V_\mu \otimes P)}}^{\dagger}_{DC} \delta_{c_4 c_1} \delta_{c_3 c_2}$$

$$\hat{P}_2 = \frac{1}{N} \overline{\overline{(V_\mu \otimes P)}}^{\dagger}_{BA} \overline{\overline{(V_\mu \otimes P)}}^{\dagger}_{DC} \delta_{c_2 c_1} \delta_{c_4 c_3}$$

$$\hat{P}_3 = \frac{1}{N} \overline{\overline{(A_\mu \otimes P)}}^{\dagger}_{BA} \overline{\overline{(A_\mu \otimes P)}}^{\dagger}_{DC} \delta_{c_4 c_1} \delta_{c_3 c_2}$$

$$\hat{P}_4 = \frac{1}{N} \overline{\overline{(A_\mu \otimes P)}}^{\dagger}_{BA} \overline{\overline{(A_\mu \otimes P)}}^{\dagger}_{DC} \delta_{c_2 c_1} \delta_{c_4 c_3}$$

RI-MOM scheme

We apply the projection operator to tree level amputated Green's function.

$$\mathrm{tr}[\Lambda^{\text{tree}}_{B_K}\hat{P}_1] = \mathrm{tr}[\Lambda^{\text{tree}}_{B_K}\hat{P}_2] = \mathrm{tr}[\Lambda^{\text{tree}}_{B_K}\hat{P}_3] = \mathrm{tr}[\Lambda^{\text{tree}}_{B_K}\hat{P}_4] = 1$$
$$\mathrm{tr}[\Lambda^{\text{tree}}_{B_K}\hat{P}_{(D)}] = \mathrm{tr}[\Lambda^{\text{tree}}_{(D)}\hat{P}_{(C)}] = 0\,,$$

The RI-MOM scheme prescription is

$$\mathrm{tr}[\Lambda^R(\widetilde{p},\widetilde{p},\widetilde{p},\widetilde{p})\hat{P}] = \mathrm{tr}[\Lambda^{\text{tree}}(\widetilde{p},\widetilde{p},\widetilde{p},\widetilde{p})\hat{P}]$$

Therefore,

$$\mathrm{tr}[\Lambda^R_{B_K}\hat{P}_1] = \mathrm{tr}[\Lambda^R_{B_K}\hat{P}_2] = \mathrm{tr}[\Lambda^R_{B_K}\hat{P}_3] = \mathrm{tr}[\Lambda^R_{B_K}\hat{P}_4] = 1$$
$$\mathrm{tr}[\Lambda^R_{B_K}\hat{P}_{(D)}] = \mathrm{tr}[\Lambda^R_{(D)}\hat{P}_{(C)}] = 0$$

Here,
(C): $\{O_1, O_2, O_3, O_4\}$ (diagonal operators)
(D): remaining operators (off-diagonal operators)

We define the projected amputated Green's function as follows.

$$\Gamma^B_{\alpha\beta} \equiv \frac{1}{z_q^2} \mathrm{tr}[\Lambda^B_\alpha \hat{P}_\beta]$$

where α, β represent operators and z_q is obtained from conserved vector current channel.

$$1 = z_1 \Gamma^B_{1\alpha} + z_2 \Gamma^B_{2\alpha} + z_3 \Gamma^B_{3\alpha} + z_4 \Gamma^B_{4\alpha} + \sum_{\gamma \in (D)} z_\gamma \Gamma^B_{\gamma\alpha}\,, \quad \alpha \in (C)$$

$$0 = z_1 \Gamma^B_{1\beta} + z_2 \Gamma^B_{2\beta} + z_3 \Gamma^B_{3\beta} + z_4 \Gamma^B_{4\beta} + \sum_{\gamma \in (D)} z_\gamma \Gamma^B_{\gamma\beta}\,, \quad \beta \in (D)$$

We can express these equations as a matrix equation.

$$\vec{z}_{\text{tree}} = \vec{z} \cdot \hat{\Gamma}^B ,$$

where

$$\vec{z}_{\text{tree}} = (1, 1, 1, 1, 0, \cdots, 0)$$
$$\vec{z} = (z_1, z_2, z_3, z_4, z_5, z_6, \cdots)$$

$$\hat{\Gamma}^B = \left(\begin{array}{cccc|ccc} \Gamma^B_{11} & \Gamma^B_{12} & \Gamma^B_{13} & \Gamma^B_{14} & \Gamma^B_{15} & \Gamma^B_{16} & \cdots \\ \Gamma^B_{21} & \Gamma^B_{22} & \Gamma^B_{23} & \Gamma^B_{24} & \Gamma^B_{25} & \Gamma^B_{26} & \cdots \\ \Gamma^B_{31} & \Gamma^B_{32} & \Gamma^B_{33} & \Gamma^B_{34} & \Gamma^B_{35} & \Gamma^B_{36} & \cdots \\ \Gamma^B_{41} & \Gamma^B_{42} & \Gamma^B_{43} & \Gamma^B_{44} & \Gamma^B_{45} & \Gamma^B_{46} & \cdots \\ \hline \Gamma^B_{51} & \Gamma^B_{52} & \Gamma^B_{53} & \Gamma^B_{54} & \Gamma^B_{55} & \Gamma^B_{56} & \cdots \\ \Gamma^B_{61} & \Gamma^B_{62} & \Gamma^B_{63} & \Gamma^B_{64} & \Gamma^B_{65} & \Gamma^B_{66} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

Then, we can obtain the $\vec{z}$ from the following equation.

$$\vec{z} = \vec{z}_{\text{tree}} \cdot (\hat{\Gamma}^B)^{-1}$$

The $\hat{\Gamma}^B$ can be rewritten by sub-matrices as follows.

$$\hat{\Gamma}^B = \begin{pmatrix} X_{4\times4} & Y_{4\times20} \\ Z_{20\times4} & W_{20\times20} \end{pmatrix}$$

Here, X is diagonal terms.

$$X = \begin{pmatrix} \Gamma^B_{11} & \Gamma^B_{12} & \Gamma^B_{13} & \Gamma^B_{14} \\ \Gamma^B_{21} & \Gamma^B_{22} & \Gamma^B_{23} & \Gamma^B_{24} \\ \Gamma^B_{31} & \Gamma^B_{32} & \Gamma^B_{33} & \Gamma^B_{34} \\ \Gamma^B_{41} & \Gamma^B_{42} & \Gamma^B_{43} & \Gamma^B_{44} \end{pmatrix}$$

Y is off-diagonal terms.

$$Y = \begin{pmatrix} \Gamma^B_{15} & \Gamma^B_{16} & \cdots \\ \Gamma^B_{25} & \Gamma^B_{26} & \cdots \\ \Gamma^B_{35} & \Gamma^B_{36} & \cdots \\ \Gamma^B_{45} & \Gamma^B_{46} & \cdots \end{pmatrix}$$

- The number of off-diagonal operators is 20.
- $\{O_5, O_6, \cdots\} = \{(S\otimes V)(S\otimes V)_{\mathbf{I}}, (S\otimes V)(S\otimes V)_{\mathbf{II}}, (S\otimes A)(S\otimes A)_{\mathbf{I}}, \cdots\}$.
- We assume that $Z \simeq Y^T \simeq \mathcal{O}(\alpha_s)$. It is an approximation within factor of 2.
- $W \simeq 1 + \mathcal{O}(\alpha_s)$.

The inverse of block matrix is

$$(\Gamma^B)^{-1} = \begin{pmatrix} (X - YW^{-1}Z)^{-1} & -X^{-1}Y(W - ZX^{-1}Y)^{-1} \\ -W^{-1}Z(X - YW^{-1}Z)^{-1} & (W - ZX^{-1}Y)^{-1} \end{pmatrix}$$

Using power series expansion in Y and Z,

$$(\Gamma^B)^{-1} \simeq \begin{pmatrix} X^{-1} + X^{-1}YW^{-1}ZX^{-1} & -X^{-1}Y(W^{-1} + W^{-1}ZX^{-1}YW^{-1}) \\ -W^{-1}Z(X^{-1} + X^{-1}YW^{-1}ZX^{-1}) & W^{-1} + W^{-1}ZX^{-1}YW^{-1} \end{pmatrix}$$

With our assumption, $Z \simeq Y^T$ and $W \simeq 1$

$$(\Gamma^B)^{-1} \simeq \begin{pmatrix} X^{-1} + X^{-1}YY^TX^{-1} & -X^{-1}Y(1 + Y^TX^{-1}Y) \\ -Y^T(X^{-1} + X^{-1}YY^TX^{-1}) & 1 + Y^TX^{-1}Y \end{pmatrix}$$

- X^{-1}: central value of z-factors.
- $X^{-1}YY^TX^{-1} \sim O(\alpha_s^2)$: diagonal correction terms.
- $X^{-1}Y \sim O(\alpha_s)$: off-diagonal correction terms.

Simulation Detail

- $20^3 \times 64$ MILC asqtad lattice ($a \approx 0.12 fm,\ am_\ell/am_s = 0.01/0.05$).
- HYP smeared staggered fermions as valence quarks.
- The number of configurations is 30.
- 5 valence quark masses (0.01, 0.02, 0.03, 0.04, 0.05)
- 9 external momenta in the units of $(\frac{2\pi}{L_s}, \frac{2\pi}{L_s}, \frac{2\pi}{L_s}, \frac{2\pi}{L_t})$.
- We use the jackknife resampling method to estimate statistical errors.

n(x, y, z, t)	$\lvert a\widetilde{p}\rvert$	GeV
$(2,2,2,7)$	1.2871	2.1332
$(2,2,2,8)$	1.3421	2.2243
$(2,2,2,9)$	1.4018	2.3233
$(2,3,2,7)$	1.4663	2.4302
$(2,3,2,8)$	1.5148	2.5106
$(2,3,2,9)$	1.5680	2.5987
$(3,2,3,8)$	1.6698	2.7674
$(3,3,3,7)$	1.7712	2.9355
$(3,3,3,9)$	1.8562	3.0764

Data analysis

$$\Gamma^B = \begin{pmatrix} X_{4\times4} & Y_{4\times20} \\ Z_{20\times4} & W_{20\times20} \end{pmatrix}$$

As an example, we consider the data analysis of Γ^B_{11}.

$$X = \begin{pmatrix} \Gamma^B_{11} & \Gamma^B_{12} & \Gamma^B_{13} & \Gamma^B_{14} \\ \Gamma^B_{21} & \Gamma^B_{22} & \Gamma^B_{23} & \Gamma^B_{24} \\ \Gamma^B_{31} & \Gamma^B_{32} & \Gamma^B_{33} & \Gamma^B_{34} \\ \Gamma^B_{41} & \Gamma^B_{42} & \Gamma^B_{43} & \Gamma^B_{44} \end{pmatrix}$$

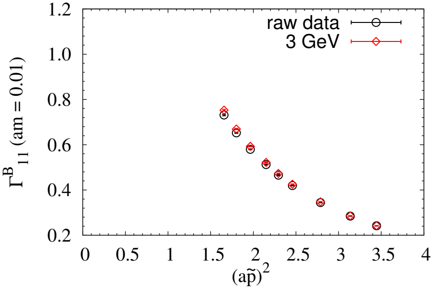
Γ_{11}^B analysis

We convert the scale of raw data in the RI-MOM scheme from $\mu(= |\widetilde{p}|)$ to the common scale $\mu_0 = 3\text{GeV}$ using two-loop RG evolution factor $U_{B_K}^{\text{RI-MOM}}(\mu_0, \mu)$.

$$\vec{z}^{\,\text{RI-MOM}}(\mu_0) = U_{B_K}^{\text{RI-MOM}}(\mu_0, \mu)\ \vec{z}^{\,\text{RI-MOM}}(\mu)$$

$$\vec{z}^{\,\text{RI-MOM}} = \vec{z}_{\text{tree}}(\Gamma^B)^{-1}$$

$$\Gamma_{11}^B(\mu_0) = 1/U_{B_K}^{\text{RI-MOM}}(\mu_0, \mu)\Gamma_{11}^B(\mu)$$



m-fit (fitting with respect to quark mass)

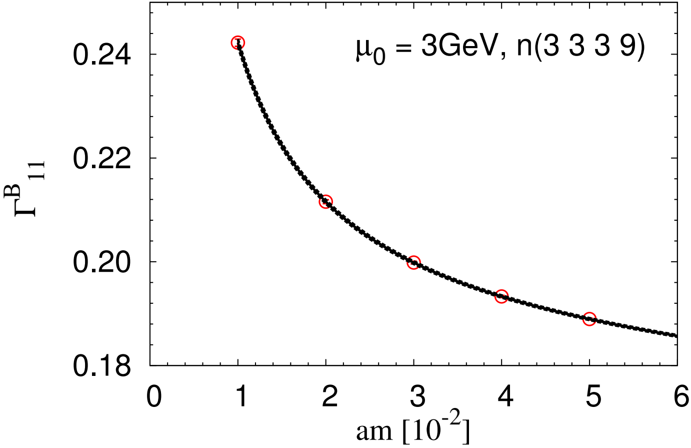
We fit the data with respect to quark mass for a fixed momentum to the following function f. [RBC, PRD66, 2002]

$$f(m, a, \tilde{p}) = c_1 + c_2 \cdot am + c_3 \cdot \frac{1}{(am)} + c_4 \cdot \frac{1}{(am)^2},$$

After m-fit, we take the $c_1(a\widetilde{p})$ as chiral limit values. The sea quark determinant contributions($c_3, c_4 \propto (m_\ell^2 m_s)^\nu$) with ν the number of zero modes, these pole terms contributions vanish in the chiral limit.

μ_0	c_1	c_2	c_3	c_4	χ^2/dof
3GeV	0.17991(20)	-0.0975(15)	0.0007118(85)	-0.000000790(36)	0.00194(40)

m-fit plot



m-fit $(\mu_0 = 3\text{GeV})$

p-fit (fitting with respect to reduced momentum)

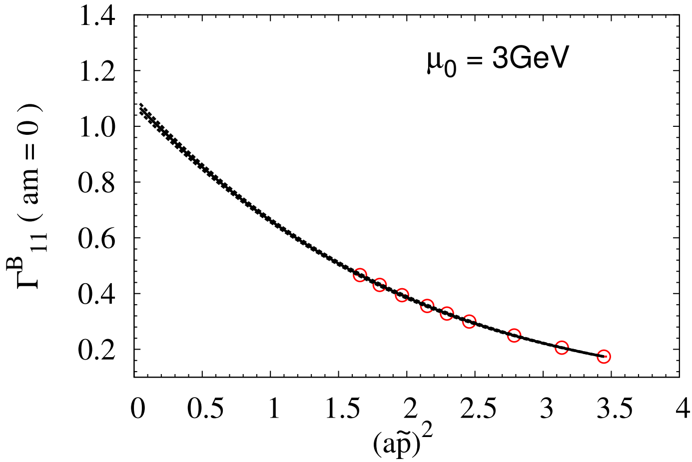
We fit $c_1(a\widetilde{p})$ to the following fitting function.

$$g(a\widetilde{p}) = b_1 + b_2 \cdot (a\widetilde{p})^2 + b_3 \cdot ((a\widetilde{p})^2)^2 + b_4 \cdot (a\widetilde{p})^4 + b_5 \cdot ((a\widetilde{p})^2)^3$$

To avoid non-perturbative effects at small $(a\widetilde{p})^2$, we choose the momentum window as $(a\widetilde{p})^2 > 1$. Because we assume that those terms of $\mathcal{O}((a\widetilde{p})^2)$ and higher order are pure lattice artifacts, we take the b_1 as Γ_{11}^B value at $\mu_0 = 3\text{GeV}$ in the RI-MOM scheme.

μ_0	b_1	b_2	b_3	b_4	b_5	χ^2/dof
3GeV	1.088(16)	-0.515(18)	0.0953(74)	0.0020(65)	-0.00663(91)	0.08(17)

p-fit plot



p-fit ($\mu_0 = 3\text{GeV}$)

Γ^{-1}(3GeV) matrix result

$$(\Gamma^B)^{-1} = \begin{pmatrix} X^{-1} + X^{-1}YY^TX^{-1} & -X^{-1}Y(1+Y^TX^{-1}Y) \\ -Y^T(X^{-1}+X^{-1}YY^TX^{-1}) & 1+Y^TX^{-1}Y \end{pmatrix}$$

$$X^{-1} = \begin{pmatrix} 1.333(32) & -0.793(45) & 0.336(21) & 0.007(31) \\ -0.726(44) & 1.940(41) & -0.008(30) & -0.047(33) \\ 0.341(30) & 0.032(45) & 1.222(32) & -0.604(37) \\ 0.018(45) & -0.084(52) & -0.637(39) & 1.543(36) \end{pmatrix}$$

$$X^{-1}YY^TX^{-1} = \begin{pmatrix} 0.0127(15) & -0.0079(11) & 0.0020(17) & -0.0001(10) \\ -0.00713(92) & 0.0064(11) & -0.0010(11) & -0.00045(73) \\ 0.0020(18) & 0.0000(15) & 0.0200(90) & -0.0103(48) \\ 0.0001(12) & -0.0012(11) & -0.0109(52) & 0.0059(28) \end{pmatrix}$$

- X^{-1} : central value of z-factors
- $X^{-1}YY^TX^{-1}$: diagonal correction terms, systematic error of z-factors.
- $-X^{-1}Y \lesssim 7\%$: off-diagonal correction terms, their effect becomes $\ll 0.07\%$ by wrong taste suppression. We neglect them.

Result of renormalization factors of diagonal operators

We obtain $\vec{z}$ in RI-MOM scheme at 3GeV.

$$\vec{z} = \vec{z}_{\text{tree}} \cdot (\hat{\Gamma}^B)^{-1} ,$$

where

$$\vec{z}_{\text{tree}} = (1, 1, 1, 1, 0, \cdots, 0)$$
$$\vec{z} = (z_1, z_2, z_3, z_4, z_5, z_6, \cdots)$$

We convert the scheme from RI-MOM $\to \overline{\text{MS}}$ using two-loop RG evolution factor.

	RI-MOM(3GeV)	$\overline{\text{MS}}$(3GeV)
z_1	0.9666(78)	0.9812(79)
z_2	1.095(30)	1.111(31)
z_3	0.9139(73)	0.9277(74)
z_4	0.898(28)	0.912(29)

Systematic Error

We estimate systematic errors.

- The first systematic error comes from diagonal correction terms.
  $E_{diag} \equiv \vec{z}_{\text{tree}} \cdot X^{-1} Y Y^T X^{-1}$
- The second systematic error comes from off-diagonal correction terms. Their size $(-X^{-1}Y)$ are typically less than 7%. By the wrong taste suppression($\ll 1\%$), their effect becomes $\ll 0.07\%$. Hence, we neglect them.
- Another systematic error comes from truncated higher order of the two-loop RG evolution factor (RI-MOM$\rightarrow \overline{\text{MS}}$) : $E_t = z_i \cdot \alpha_s^3$
- We add these systematic errors in quadrature: $E_{tot} = \sqrt{E_{diag}^2 + E_t^2}$.

	$\overline{\text{MS}}$(3GeV)	E_{diag}	E_t	E_{tot}
z_1	0.9812(79)	0.0077	0.0144	0.0163
z_2	1.111(31)	0.0027	0.0163	0.0165
z_3	0.9277(74)	0.0101	0.0136	0.0171
z_4	0.912(29)	0.0050	0.0134	0.0143

Compare with one-loop matching factor

- We compare the NPR result($\overline{\mathrm{MS}}$ [NDR]) with those of one-loop perturbative matching.
- We quote truncated two-loop uncertainty : $E_t^{\text{one-loop}} \equiv z_i \cdot \alpha_s^2$ as our estimate of the systematic error of one-loop matching.

	NPR(3GeV)	one-loop(3GeV)	Δ
z_1	0.981(8)(16)	1.035(62)	0.83 σ
z_2	1.111(31)(17)	1.120(67)	0.12 σ
z_3	0.928(7)(17)	1.043(63)	1.75 σ
z_4	0.912(29)(14)	0.953(57)	0.63 σ

- The result of NPR are consistent with those of one-loop matching within 2σ. This indicates that out NPR results are quite reasonable.

Summary

- We compute the matching factor for B_K operator using NPR method in the RI-MOM scheme.
- We compare the NPR result with those of one-loop perturbative matching.
- The NPR results are quite reasonable.
- We plan to analyse the BSM operators in near future.